NC STATE UNIVERSITY

Semidiscrete Central-Upwind Schemes for Hyperbolic Conservation Laws

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Finite-Volume Schemes for Conservation Laws

Goal: Solve general multi-D system of conservation laws:

$$u_t + \nabla_x \cdot f(u) = 0, \quad x \in \mathbb{R}^d$$

- Notation: $x_j := x_l + (j + 0.5)\Delta x$, $x_{j\pm\frac{1}{2}} := x_j \pm \Delta x/2, t^n := \sum_{k=1}^{n} \Delta t^k$, and $u_j^n := u(x_j, t^n)$
- Define cell averages: $\overline{u}(x,t) := \frac{1}{\Delta x} \int_{I(x)} u(\xi, t)\, d\xi, \quad I(x) = \left\{\xi : |\xi - x| < \frac{\Delta x}{2}\right\}$
- Reformulate (1) using cell averages:

$$\overline{u}(x, t + \Delta t) = \overline{u}(x,t) - \frac{1}{\Delta x}\left[\int_{\tau=t}^{t+\Delta t} f\left(u\left(x + \frac{\Delta x}{2}, \tau\right)\right) d\tau - \int_{\tau=t}^{t+\Delta t} f\left(u\left(x - \frac{\Delta x}{2}, \tau\right)\right) d\tau\right] \quad (1)$$

Godunov-Type Central Schemes (1D)[1]

- At time t^n, assume we have piecewise polynomial: $\tilde{u}(x, t^n) = p_j^n(x), \quad x \in \left(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right)$
- Evolve $\tilde{u}(x, t^n)$ according to (1) - this defines the FV update for each cell
- If we set $x = x_{j+\frac{1}{2}}$, we obtain the Godunov-type central scheme:

$$
\begin{aligned}
\overline{u}_{j+\frac{1}{2}}^{n+1} = \frac{1}{\Delta x}\left[\int_{x_j}^{x_{j+\frac{1}{2}}} p_j^n(x)\,dx + \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} p_{j+1}^n(x)\,dx\right] \\
- \frac{\lambda}{\Delta t}\left[\int_{t^n}^{t^{n+1}} f(u(x_{j+1}, t))\,dt - \int_{t^n}^{t^{n+1}} f(u(x_j, t))\,dt\right], \quad \lambda := \frac{\Delta t^n}{\Delta x}
\end{aligned}
$$

- Solution is smooth in neighborhoods of $\{x_j\}$, so flux integrals can be discretized

[1] Godunov, S. K. 1959 *Мат. сборник* 47 (89).

Central-Upwind Schemes (1D)

- Estimate one-sided propagation speeds of discontinuities located at $\left\{x_{j\pm\frac{1}{2}}\right\}$:

$$a^+_{j+\frac{1}{2}} := \max_{\omega\in C\left(u^-_{j+\frac{1}{2}},u^+_{j+\frac{1}{2}}\right)} \left\{\lambda_N\left(\frac{\partial f}{\partial u}(\omega)\right),0\right\} \qquad a^-_{j+\frac{1}{2}} := \min_{\omega\in C\left(u^-_{j+\frac{1}{2}},u^+_{j+\frac{1}{2}}\right)} \left\{\lambda_1\left(\frac{\partial f}{\partial u}(\omega)\right),0\right\}$$

- Use these speeds to define *different* rectangular domains:

$$D_{\text{int}} := \left[x^n_{j-\frac{1}{2},r}, x^n_{j+\frac{1}{2},l}\right] \times \left[t^n, t^{n+1}\right] \text{ and } D_{\text{fan}} := \left[x^n_{j+\frac{1}{2},l}, x^n_{j+\frac{1}{2},r}\right] \times \left[t^n, t^{n+1}\right]$$

$$x^n_{j+\frac{1}{2},l} := x_{j+\frac{1}{2}} + \Delta t^n a^-_{j+\frac{1}{2}} \text{ and } x^n_{j+\frac{1}{2},r} := x_{j+\frac{1}{2}} + \Delta t^n a^+_{j+\frac{1}{2}}$$

Central-Upwind Schemes (1D)[2]

- Integrate conservation law over *each* domain to produce two new cell averages:

$$\overline{w}_j^{n+1} = \frac{1}{x^n_{j+\frac{1}{2},l} - x^n_{j-\frac{1}{2},r}} \left[\int_{x^n_{j-\frac{1}{2},r}}^{x^n_{j+\frac{1}{2},l}} p_j^n(x)\,dx - \int_{t^n}^{t^{n+1}} f\left(u\left(x^n_{j+\frac{1}{2},l}, t\right)\right) - f\left(u\left(x_{j-\frac{1}{2},r}, t\right)\right)\,dt \right] \text{(on } D_{\text{int}}\text{)}$$

$$\overline{w}_{j+\frac{1}{2}}^{n+1} = \frac{1}{x^n_{j+\frac{1}{2},r} - x^n_{j+\frac{1}{2},l}} \Bigg[\int_{x_{j+\frac{1}{2},l}}^{x_{j+\frac{1}{2}}} p_j^n(x)\,dx + \int_{x_{j+\frac{1}{2}}}^{x^n_{j+\frac{1}{2},r}} p_{j+1}^n(x)\,dx - \int_{t^n}^{t^{n+1}} f\left(u\left(x^n_{j+\frac{1}{2},r}, t\right)\right) - f\left(u\left(x^n_{j+\frac{1}{2},l}, t\right)\right)\,dt \Bigg] \quad \text{(on } D_{\text{fan}}\text{)}$$

- Construct staggered piecewise polynomial $\tilde{w}^{n+1}(x)$, and project back onto original grid

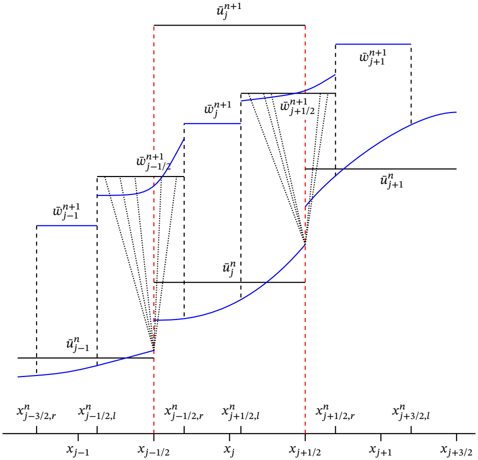
[2]Kurganov, A., et al. 2001 *SIAM Journal on Scientific Computing* 23 (3).

Central-Upwind Scheme in Conservative Form

- Can express scheme as:

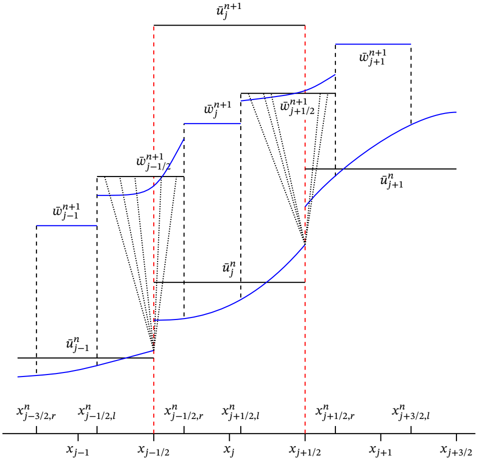
$$\frac{d}{dt}\overline{u}_j(t) = -\frac{H_{j+\frac{1}{2}}(t) - H_{j-\frac{1}{2}}(t)}{\Delta x}$$

$$H_{j+\frac{1}{2}}(t) := \frac{a^+_{j+\frac{1}{2}} f(u^-_{j+\frac{1}{2}}) - a^-_{j+\frac{1}{2}} f(u^+_{j+\frac{1}{2}})}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + \frac{a^+_{j+\frac{1}{2}} a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\left[u^+_{j+\frac{1}{2}} - u^-_{j+\frac{1}{2}}\right]$$



Central-Upwind Schemes Advantages

- Utilizing eigenvalues to calculate propagation speed is more accurate, reduces numerical dissipation.
- No required Riemann solvers or characteristic decompositions.
- Utilizing one-sided local speeds allows for upwinding.



Implementation Details

- Second order SSP RK time integration.

$$\overline{u}^{(1)} = \overline{u}^{n} + \Delta t\, \mathcal{L}\left(\overline{u}^{n}\right)$$
$$\overline{u}^{n+1} = \frac{1}{2}\overline{u}^{n} + \frac{1}{2}\overline{u}^{(1)} + \frac{1}{2}\Delta t\, \mathcal{L}\left(\overline{u}^{(1)}\right)$$

- CFL based time stepping.

$$\Delta t^{n+1} = C_{\text{CFL}} \min_{\Omega}\left(\min_{d}\left(\frac{\Delta_d}{s_d}\right),\right)$$

- Generalized minmod reconstruction

$$\sigma_{j,d}^{n} = \text{minmod}\left(\theta\frac{\overline{u}_j^{n} - \overline{u}_{j-1}^{n}}{\Delta_d}, \frac{\overline{u}_{j+1}^{n} - \overline{u}_{j-1}^{n}}{2\Delta_d}, \theta\frac{\overline{u}_{j+1}^{n} - \overline{u}_j^{n}}{\Delta_d}\right)$$

- For all cases $\theta = 2$

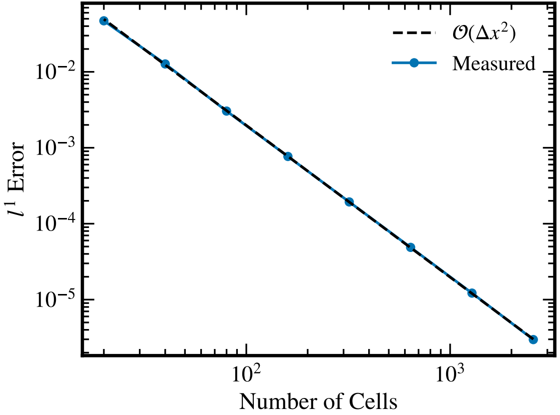
Burgers' Equations

$$u_t + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0$$

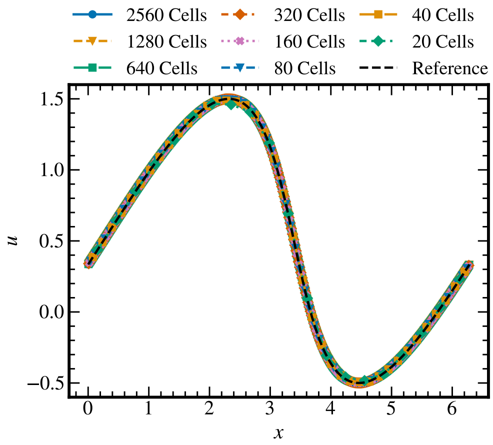
- Example 1 presented by Kurganov et. al.[2]

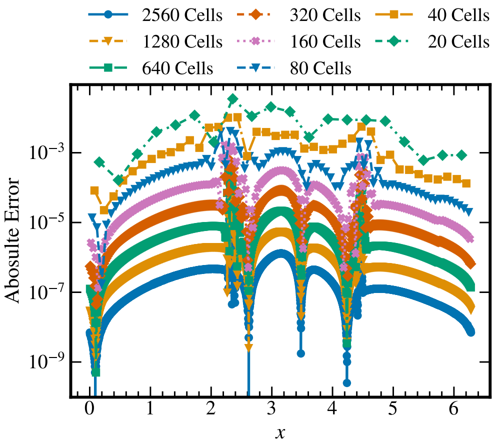
$$u(x,0) = 0.5 + \sin(x)$$
$$x \in [0, 2\pi]$$

- Periodic boundary conditions
- Simulated until $t = 0.5$.
- Compared to a reference solution computed with 10240 cells.



Solutions and Errors





Euler Equations

One Dimensional

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ \rho u \\ E\end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix}\rho u \\ \rho u^2 + p \\ u\left(E + p\right)\end{pmatrix} = 0$$

$$p = (1-\gamma)\left[E - \frac{\rho}{2}u^2\right]$$

Two Dimensional

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ \rho u \\ \rho v \\ E\end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix}\rho u \\ \rho u^2 + p \\ \rho u v \\ u\left(E + p\right)\end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix}\rho u \\ \rho u v \\ \rho v^2 + p \\ v\left(E + p\right)\end{pmatrix} = 0$$

$$p = (1-\gamma)\left[E - \frac{\rho}{2}\left(u^2 + v^2\right)\right]$$

1D Euler Equations: Advection of Smooth Density

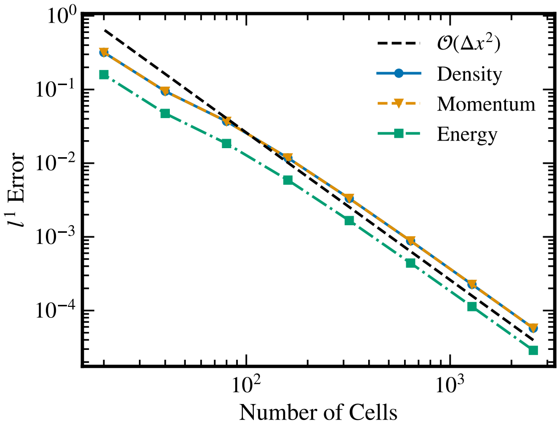
- Section 5.1.1 presented by Micalizzi and Toro.[3]

$$\mathbf{u}(x,0) = \begin{cases} \rho(x,0) = 2 + \sin^4(\pi x) \\ u(x,0) = u_\infty = 1 \\ p(x,0) = p_\infty = 1 \end{cases}$$

$$x \in [-1, 1]$$

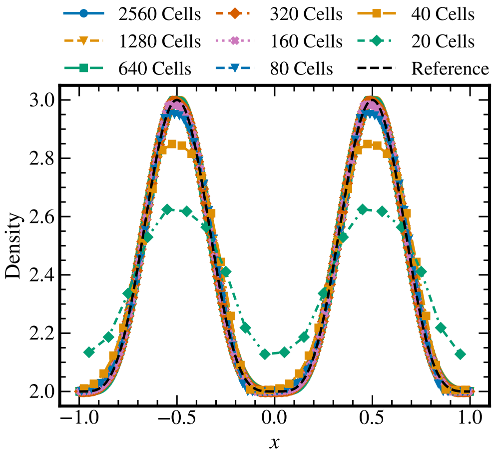
- Periodic boundary conditions
- Simulated until $t = 2.0$.
- Compared to the exact solution

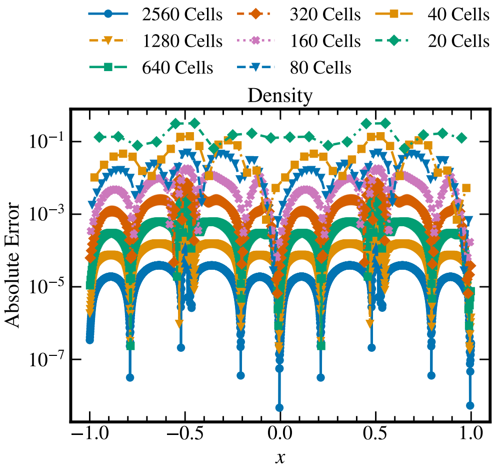
$$\mathbf{u}(x,t) = \mathbf{u}(x - t, 0)$$



[3]Micalizzi, L., et al. 2024 *arXiv preprint arXiv:2411.07422*.

Solution and Errors





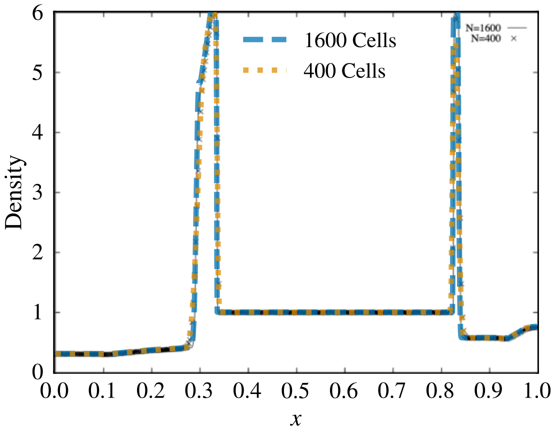
1D Euler Equations: Blast Wave

- Example 2 presented by Kurganov et. al.[2]

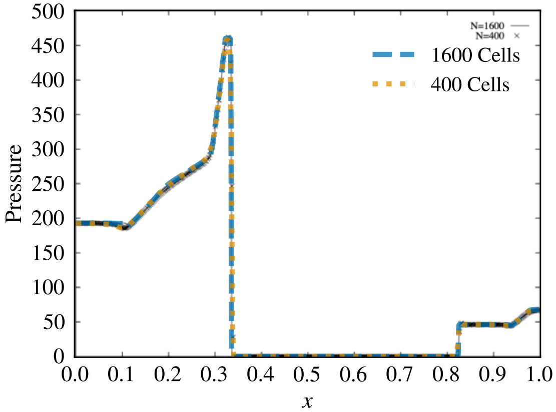
$$\mathbf{u}(x,0) = \begin{cases} [1,0,2500]^T & \text{if } 0 \le x < 0.1 \\ [1,0,0.025]^T & \text{if } 0.1 \le x < 0.9 \\ [1,0,250]^T & \text{if } 0.9 \le x < 1 \end{cases}$$

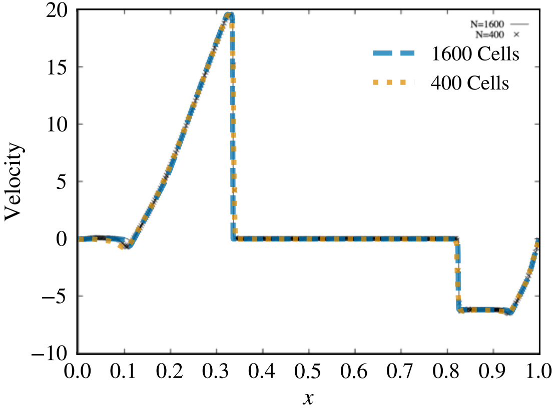
$$x \in [0,1]$$

- Solid wall boundary conditions.
- Simulated until $t = 0.01$.
- Solutions compared to Kurganov et. al.
- Required positive pressure fix.



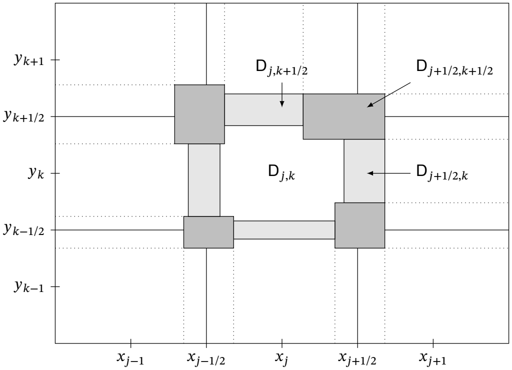
1D Euler Equations: Blast Wave Cont.





Generalization to Higher Dimensions (2D)

- Two fluxes: $f(x), g(y)$ - requires 2D CFL condition
- Smooth solution in D_{jk}, non-smooth along fans propagating from **4** cell interfaces
- Implemented conservative form - still component-wise (doesn't require Riemann solvers)
- Generalization to Hamilton Jacobi, Advection-Convection, and $d \geq 3$



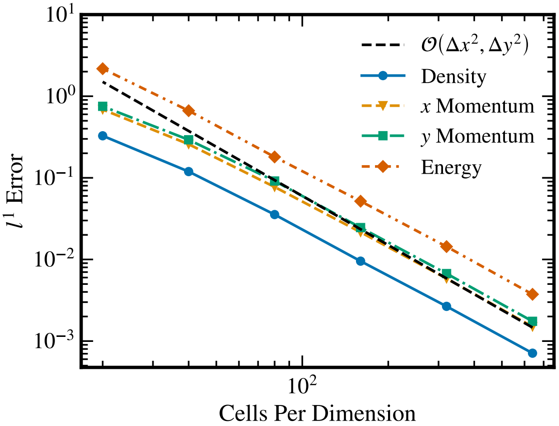
2D Euler Equations: Smooth Isentropic Unsteady Vortex

- Example 5.2.1 presented by Micalizzi and Toro.[3]

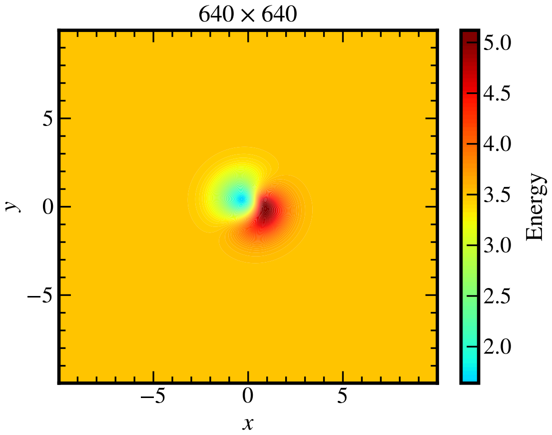
$$\begin{bmatrix} \rho(\vec{r},0) \\ \rho \begin{bmatrix} u(\vec{r},0) \\ v(\vec{r},0) \end{bmatrix} \\ E(\vec{r},0) \end{bmatrix} = \begin{bmatrix} \rho(\vec{r},0) = (1+\delta T)^{(\gamma-1)^{-1}} \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{\beta}{2\pi} \exp\left(\frac{1-r^2}{2}\right) \begin{bmatrix} -y \\ x \end{bmatrix} \\ p(\vec{r},0) = (1+\delta T)^{\gamma(\gamma-1)^{-1}} \end{bmatrix},$$

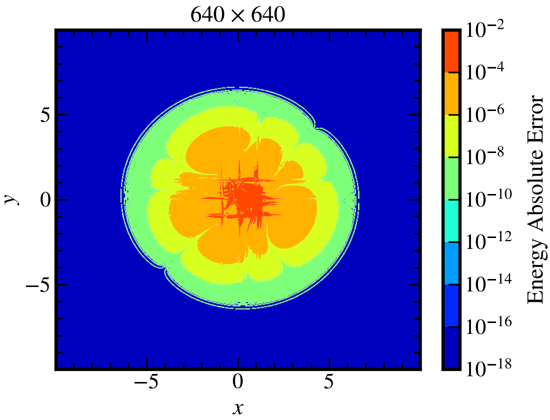
$$\delta T = -\frac{(\gamma-1)\beta^2}{8\gamma\pi^2} e^{1-r^2} \quad r = \sqrt{x^2+y^2}$$

- Periodic boundary conditions.
- Simulated until $t = 0.1$.
- Compared to reference solution computed on a 2560×2560 grid.



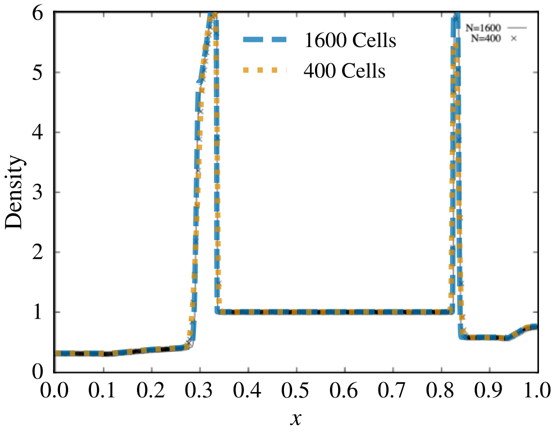
Solution and Errors



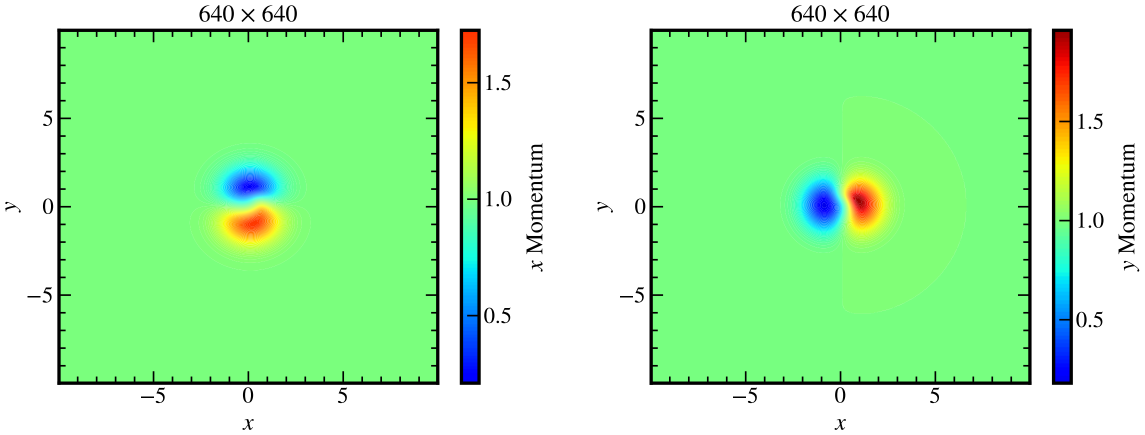


Conclusions

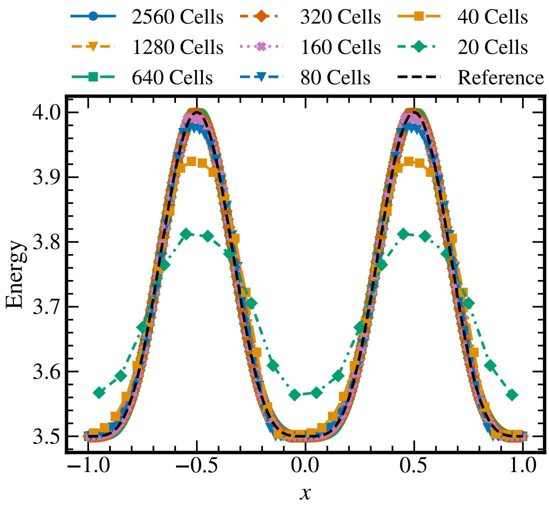
- All smooth test cases demonstrate second order accuracy.
- Our implementation is able to replicate the blast wave case.
- The linear reconstruction scheme can struggle when there are significant discontinuities.
  - This issue can be exacerbated in systems whose variables vary across orders of magnitude.



Thank You For Listening!



1. Godunov, S. K. 1959 “Разностнй метод численного ресчета разрывниh решений уравнений гидродинамики” *Мат. сборник* 47 (89),
2. Kurganov, A., S. Noelle, and G. Petrova 2001 “Semidiscrete central-upwind schemes for hyperbolic conservation laws and Hamilton–Jacobi equations” *SIAM Journal on Scientific Computing* 23 (3),
3. Micalizzi, L. and E. F. Toro 2024 “Impact of numerical fluxes on high order semidiscrete WENO-DeC finite volume schemes” *arXiv preprint arXiv:2411.07422*.


2560 Cells
320 Cells
40 Cells
1280 Cells
160 Cells
20 Cells
640 Cells
80 Cells
Reference
Energy
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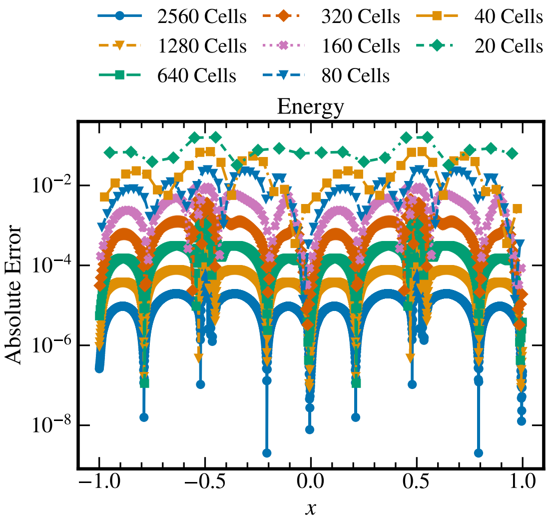
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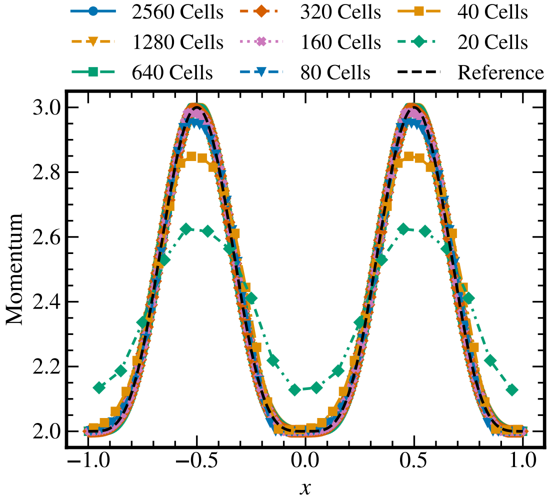
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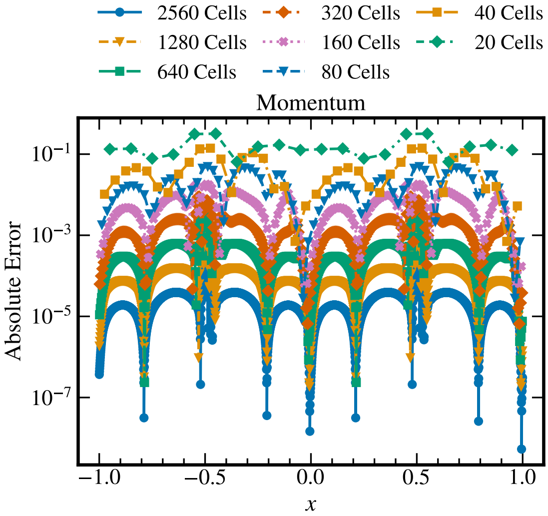
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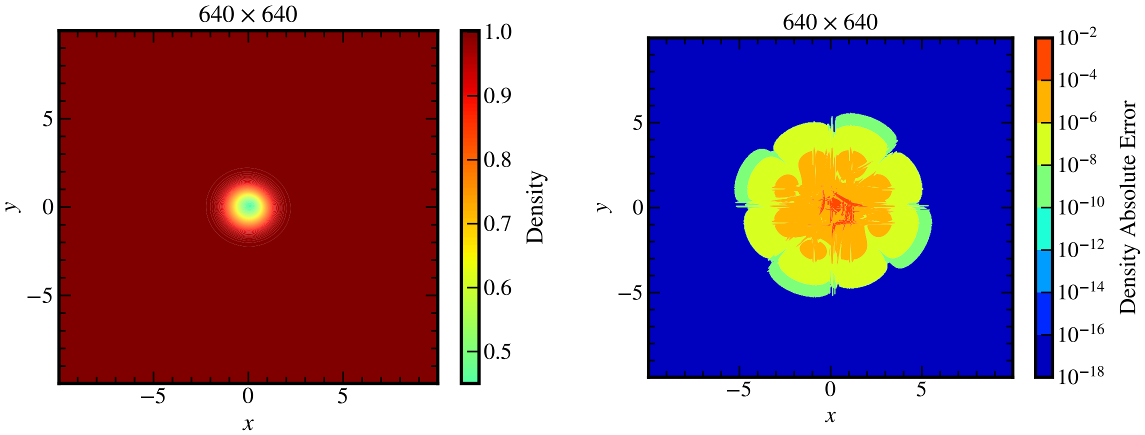
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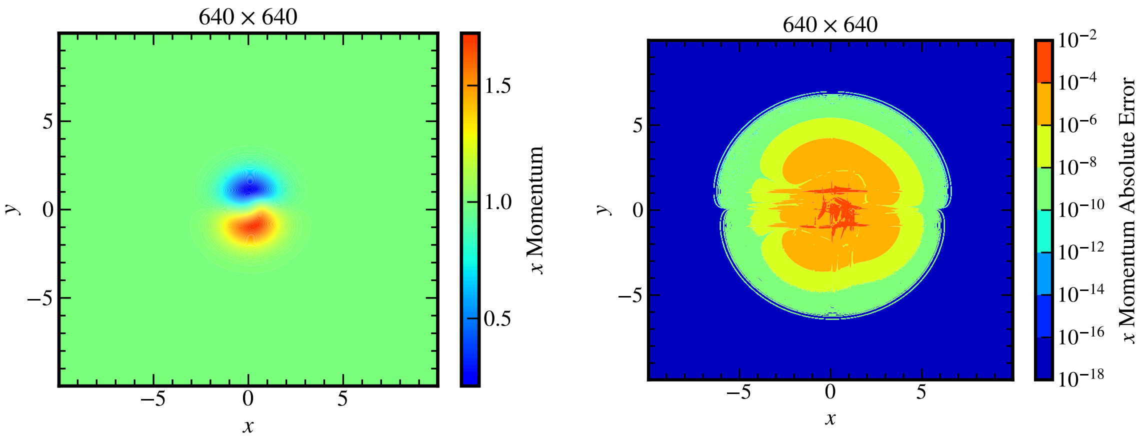
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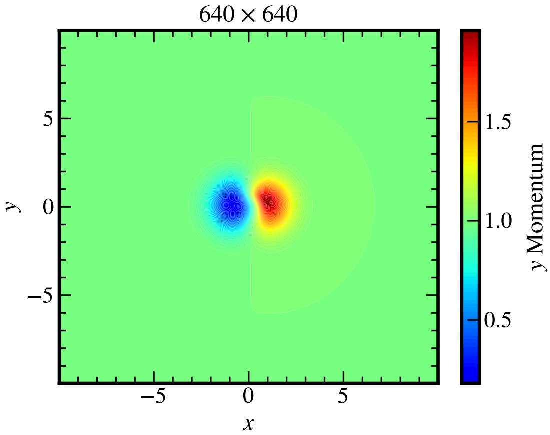
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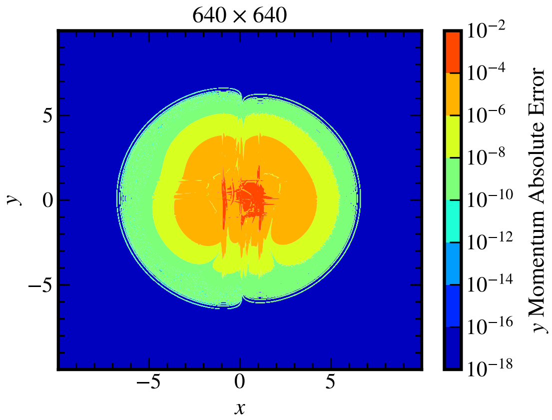
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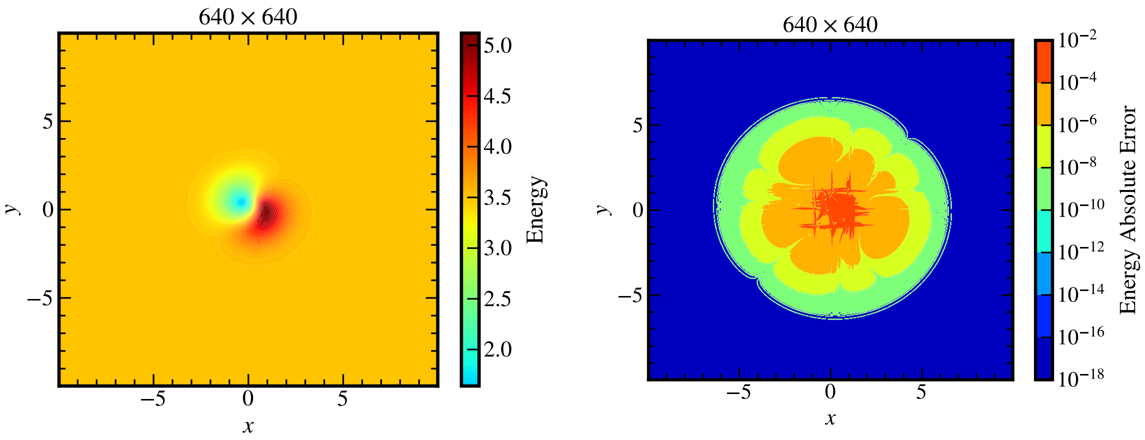
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Energy Absolute Error
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